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NONLINEAR OSCILLATIONS OF A VAPOR BUBBLE IN THE REGION OF THE MAIN RESONANCE

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Nonlinear spherical vapor bubble oscillations in the main resonance region in a viscous incompressible fluid under the action of a periodically varying external pressure are considered using the multiscale method. Saddle-node type bifurcations that appear when the amplitude or frequency of the external effect changes are investigated.

The operation of heat exchangers is substantially affected by vapor cavitation, which is observed with changes (in particular, periodic) in the pressure in the fluid. To study the processes occurring during vapor cavitation, a theory is needed that would describe the dynamics of vapor bubbles subjected to periodic changes in pressure. The present-day theory provides the most successful description of the dynamics of a radially pulsating single vapor bubble [1, 2]. However, the available works do not analyze the pattern of transition from regular to random oscillations when the external parameters of the effect exerted on a nonlinearly oscillating bubble change. On the other hand, since the oscillating bubble system is nonlinear, the effect of the parameters on the character of the oscillations is not unique, and the problem requires a separate investigation.

Earlier, the behavior of a vapor bubble was studied in the case of a resonance external effect ($\omega = \omega_0$) [3]. In the present work, using the multiscale method [4], the main resonance region $\omega \approx \omega_0$ was considered. The multiscale method was rather efficiently utilized for investigating nonlinear vibrations of gas bubbles in an incompressible fluid [5].

As a mathematical model of the pulsations of a bubble, the following equation was adopted, which takes into account the fluid viscosity [6]:

$$\rho\left[R\frac{d^2R}{d\tau^2}+\frac{3}{2}\left(\frac{dR}{d\tau}\right)^2\right]+J^2\left(\frac{1}{2\rho}-\frac{1}{\rho'}\right)-\frac{d}{d\tau}(RJ)+$$
$$+P(\infty,\ \tau)+\frac{2\sigma}{R}+\frac{4\mu}{R}\left(\frac{dR}{d\tau}-\frac{J}{\rho}\right)-P'(R,\ \tau)=0, \tag{1}$$

where

$$P(\infty,\ \tau)=P^*_\infty-P_a\cos\Omega\tau.$$

This equation describes the dynamics of a single vapor bubble in an acoustic field with account for heat and mass transfer on the bubble walls. The vapor mass flow through the cavity walls is denoted by J. The bubble itself is located in an infinite space filled with a viscous incompressible fluid. It performs spherically symmetric vibrations under the action of pressure periodically varying at infinity, $P(\infty, \tau)$.

The vapor pressure in the bubble $P'(R, \tau)$ is determined as [7]

$$P'(R,\ \tau)=P_0\left(\frac{R_0}{R}\right)^{3\gamma}(1+j_1+j_2), \tag{2}$$

where $P_0 = P^*_\infty + 2\sigma/R_0$; j_1 and j_2 are the contribution to the change in $P'(R, \tau)$ due to heat and mass transfer, respectively.

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Let us introduce the following dimensionless variables:

$$t=\Omega_0\tau;\quad \Omega=\frac{1}{R_0}\left(\frac{P_0}{\rho}\right)^{1/2};\quad \omega=\frac{\Omega}{\Omega_0};\quad W=\frac{2\sigma}{R_0P_0};$$
$$b=\frac{2\mu}{R_0(P_0\rho)^{1/2}};\quad \bar{J}=\frac{J}{\Omega_0R_0\rho};\quad \frac{P_a}{P_\infty^*}=\eta; \tag{3}$$
$$\frac{P_\infty^*}{P_0}=1-W;\quad \frac{P_a}{P_0}=\xi=(1-W)\eta.$$

The amplitude of the bubble oscillations is considered to be small. Then

$$R=R_0(1+x)=R_0(1+\varepsilon u). \tag{4}$$

The value of u is of the order of unity, and the perturbation parameter ε lies in the range $0<\varepsilon\ll 1$.

Substituting Eqs. (2), (3), and (4) into Eq. (1) we obtain the equation of the motion of the bubble walls in dimensionless form

$$\varepsilon\ddot{u}(1+\varepsilon u)+\frac{3}{2}\varepsilon^2\dot{u}^2-\varepsilon\bar{J}\dot{u}-(1+\varepsilon u)\dot{\bar{J}}+\bar{J}^2\left(\frac{1}{2}-\frac{\rho}{\rho'}\right)+$$
$$+(1-W)(1-\eta\cos\omega t)+W(1+\varepsilon u)^{-1}+2b\varepsilon(1+\varepsilon u)^{-1}\dot{u}- \tag{5}$$
$$-2bJ(1+\varepsilon u)^{-1}-(1+\varepsilon u)^{-3\gamma}(1+j_1+j_2)=0.$$

Here $\cdot = d/dt$.

To further transform Eq. (5), we represent the vapor density in the form

$$\rho'=\rho_0(1+\varepsilon u)^{-3}(1+I_1+I_2), \tag{6}$$

where I_1 and I_2 are the contribution to the change in the density due to heat and mass transfer, respectively. Then

$$J=\frac{1}{3}Z(1+\varepsilon u)^{-2}(\dot{I}_1+\dot{I}_2),\quad \rho_0/\rho=Z. \tag{7}$$

Since $x=\varepsilon u$, for the case of the main resonance we set

$$\xi=\varepsilon^3P;\quad j_1=\varepsilon^3g_1;\quad j_2=\varepsilon^3g_2;\quad I=\varepsilon^3l_1;$$
$$I_2=\varepsilon^3l_2;\quad b=\varepsilon^2B;\quad \omega=\omega_0+\varepsilon^2\delta. \tag{8}$$

Here, the parameters P, g_1, g_2, l_1, l_2, B, and δ are of the order of unity. With account for Eqs. (6)-(8), Eq. (5) becomes

$$\ddot{u}+\omega_0u=\varepsilon\left(-\frac{3}{2}\dot{u}+a_1u^2\right)+\varepsilon^2\left[\frac{3}{2}u^2u+\frac{Z}{3}(\ddot{l}_1+\ddot{l}_2)-\right.$$
$$\left.-2B\dot{u}-a_2u^3+P\cos\omega t+g_1+g_2\right]+\varepsilon^3\ldots, \tag{9}$$

where

$$\omega_0^2=3\gamma-W,\quad a_1=\frac{9}{2}\gamma(\gamma+1)-2W,\quad a_2=\frac{\gamma}{2}(9\gamma^2+18\gamma+11)-3W. \tag{10}$$

To analyze Eq. (9), we use the multiscale method, which was detailed in [4], restricting ourselves to terms of the order of smallness ε^2. According to this method, u is a function of a set of times T_n and can be presented as a series expansion in powers of ε:

$$u(t,\ \varepsilon)=u(T_0,\ T_1,\ T_2\ldots\varepsilon)=u_0(T_0,\ T_1,\ T_2\ldots)+$$

$$+\varepsilon u_1(T_0, T_1, T_2 \ldots)+\varepsilon^2 u_2(T_0, T_1, T_2 \ldots), \tag{11}$$

where $T_n = \varepsilon^n t$, n = 0, 1, 2, ... For example, the first derivatives with respect to t have the form

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \quad D_n = \frac{\partial}{\partial T_n}. \tag{12}$$

Then, using Eqs. (11), (12) and the relation

$$\omega t = \omega_0 T_0 + \delta T_2, \tag{13}$$

we group terms with the same power of ε and obtain the system of equations

$$D_0^2 u_0 + \omega_0^2 u_0 = 0, \tag{14}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_1 D_0 u_0 - \frac{3}{2}(D_0 u_0)^2 + a_1 u_0^2, \tag{15}$$

$$\begin{aligned} D_0^2 u_2 + \omega_0^2 u_2 &= -2D_1 D_0 u_1 - D_1^2 u_0 - 2D_2 D_0 u_0 - \\ &- 3D_1 u_0 D_0 u_0 - 3D_0 u_1 D_0 u_0 + 2a_1 u_1 u_0 + \frac{3}{2} u_0 (D_0 u_0)^2 + \\ &+ \frac{Z}{3}(\ddot{l}_1 + \ddot{l}_2) + P_0 \cos(\omega_0 T_0 + \delta T_2) - 2BD_0 u_0 - a_2 u_0^3 + g_1 + g_2. \end{aligned} \tag{16}$$

The solution of Eq. (14) has the form

$$u_0 = A(T_1, T_2)\exp(i\omega_0 T_0) + \bar{A}(T_1, T_2)\exp(-i\omega_0 T_0). \tag{17}$$

The functions $A(T_1, T_2)$ and $\bar{A}(T_1, T_2)$ are unknown, with the latter being the complex conjugate of $A(T_1, T_2)$.

In order to find u_1, we substitute Eq. (17) into Eq. (5). To eliminate the secular terms, we set $\partial A/\partial T_1 = 0$, i.e., $A = A(T_2)$, $\bar{A} = \bar{A}(T_2)$. Then for u_1 we have

$$\begin{aligned} u_1 = \frac{-1}{3\omega_0^2}\left(a_1 + \frac{3}{2}\omega_0^2\right)[A^2 \exp(i2\omega_0 T_0) + \bar{A}^2 \times \\ \times \exp(-i2\omega_0 T_0)] + \frac{1}{\omega_0^2} A\bar{A}(2a_1 - 3\omega_0^2). \end{aligned} \tag{18}$$

Now, we express l_1, l_2, g_1, g_2 as Fourier series:

$$l_1 = \sum_{k=-\infty}^{\infty} l_{1k} \exp[ik(\omega t + \psi_{1k})], \quad g_1 = \sum_{k=-\infty}^{\infty} g_{1k} \exp[ik(\omega t + \vartheta_{1k})]. \tag{19}$$

The expressions for l_2, g_2 are similar. Using Eqs. (17), (18), and (19), we transform Eq. (15) as

$$\begin{aligned} D_0^2 u_2 + \omega_0^2 u_2 = \Big[-i2\omega_0\left(\frac{\partial A}{\partial T_2} + BA\right) + A^2\bar{A}\left(\frac{10a_1^2}{3\omega_0^2} + \right. \\ \left. + \frac{9}{2}\omega_0^2 - 5a_1 - 3a_2\right) + \left(\frac{1}{2}P - \frac{\omega_0^2 Z}{3} l_{11} \exp i\psi_{11} - \frac{\omega_0^2 Z}{3} l_{21} \exp i\psi_{21} + \right. \\ \left. + g_{11} \exp i\vartheta_{11} + g_{21} \exp i\vartheta_{21}\right) \exp i\delta T_2\Big] \exp(i\omega_0 T_0) - \\ - \Big[A^3\left(\frac{2a_1^2}{3\omega_0^2} + \frac{9}{2}\omega_0^2 + 3a_1 + a_2\right) + \left(\frac{\omega_0^2 Z}{3} l_{13} \exp i\psi_{13} + \right. \\ \left. + \frac{\omega_0^2 Z}{3} l_{23} \exp i\psi_{23} - g_{13} \exp i\vartheta_{13} - g_{23} \exp i\vartheta_{23}\right) \times \end{aligned} \tag{20}$$

$$\times \exp(i\delta T_2)\Big]\exp(i3\omega_0 T_0) + \text{c. c.},$$

where c.c. is the corresponding complex-conjugate terms and the remaining harmonics with $k \neq \pm 1, \pm 3$.

Let

$$A = \frac{1}{2} a \exp i(\delta T_2 + \varphi). \tag{21}$$

The amplitude $a(T_2)$ and the phase shift $\varphi(T_2)$ depend on T_2. To eliminate the secular terms in Eq. (20), we equate to zero the expression in the square brackets in Eq. (20) at $\exp(i\omega_0 T_0)$. Substituting Eq. (21) into these brackets, we find an equation characterizing the interrelation between a and φ. The subsequent grouping of terms of this expression with real and imaginary parts leads to the following system of equations:

$$\begin{gathered}\frac{da}{dT_2} = -Ba - \frac{P}{2\omega_0}\sin\varphi - \frac{\omega_0 Z l_{11}}{3}\sin(\psi_{11}-\varphi) - \\ -\frac{\omega_0 Z l_{21}}{3}\sin(\psi_{21}-\varphi) + \frac{g_{11}}{\omega_0}\sin(\vartheta_{11}-\varphi) + \frac{g_{21}}{\omega_0}\sin(\vartheta_{21}-\varphi), \\ \frac{d\varphi}{dT_2} = -\frac{Na^2}{2\omega_0} - \delta - \frac{P}{2a\omega_0}\cos\varphi + \frac{1}{a}\Big[\frac{\omega_0 Z l_{11}}{3}\cos(\psi_{11}-\varphi) + \\ + \frac{\omega_0 Z l_{21}}{3}\cos(\psi_{21}-\varphi) - \frac{g_{11}}{\omega_0}\cos(\vartheta_{11}-\varphi) - \frac{g_{21}}{\omega_0}\cos(\vartheta_{21}-\varphi)\Big],\end{gathered} \tag{22}$$

where

$$N = \frac{1}{4}\left(\frac{10a_1^2}{3\omega_0^2} + \frac{9}{2}\omega_0^2 - 5a_1 - 3a_2\right). \tag{23}$$

We denote $\varepsilon a = c$ and again, according to Eq. (8), we introduce b, ξ, $\omega - \omega_0$, j_1, j_2, I_1, I_2, with the latter four variables being introduced in terms of the corresponding coefficients of series (19):

$$\begin{gathered}\frac{dc}{d\tau} = -bc - \frac{\xi}{2\omega_0}\sin\varphi + M_1\sin\varphi - M_2\cos\varphi, \\ \frac{d\varphi}{d\tau} = -\frac{Nc^2}{2\omega_0} + \omega_0 - \omega - \frac{\xi}{2c\omega_0}\cos\varphi + \frac{1}{c}(M_1\cos\varphi + M_2\sin\varphi).\end{gathered} \tag{24}$$

Here

$$\begin{gathered}M_1 = \frac{\omega_0 Z I_{11}}{3}\cos\psi_{11} + \frac{\omega_0 Z I_{21}}{3}\cos\psi_{21} - \frac{j_{11}}{\omega_0}\cos\vartheta_{11} - \frac{j_{21}}{\omega_0}\cos\vartheta_{21}, \\ M_2 = \frac{\omega_0 Z I_{11}}{3}\sin\psi_{11} + \frac{\omega_0 Z I_{21}}{3}\sin\psi_{21} - \frac{j_{11}}{\omega_0}\sin\vartheta_{11} - \frac{j_{21}}{\omega_0}\sin\vartheta_{21}.\end{gathered} \tag{25}$$

Knowing the law of change in time of the amplitude c and the phase shift φ, we write the equation of the motion of the vapor bubble walls as

$$\begin{gathered}x = \varepsilon u = \varepsilon u_0 + \varepsilon^2 u_1 = c\cos(\omega\tau + \varphi) + c^2[c_1 + c_2\cos 2(\omega\tau + \varphi)], \\ c_1 = \frac{1}{2\omega_0}\left(a_1 - \frac{3}{2}\omega_0^2\right), \quad c_2 = -\frac{1}{6\omega_0^2}\left(a_1 + \frac{3}{2}\omega_0^2\right).\end{gathered} \tag{26}$$

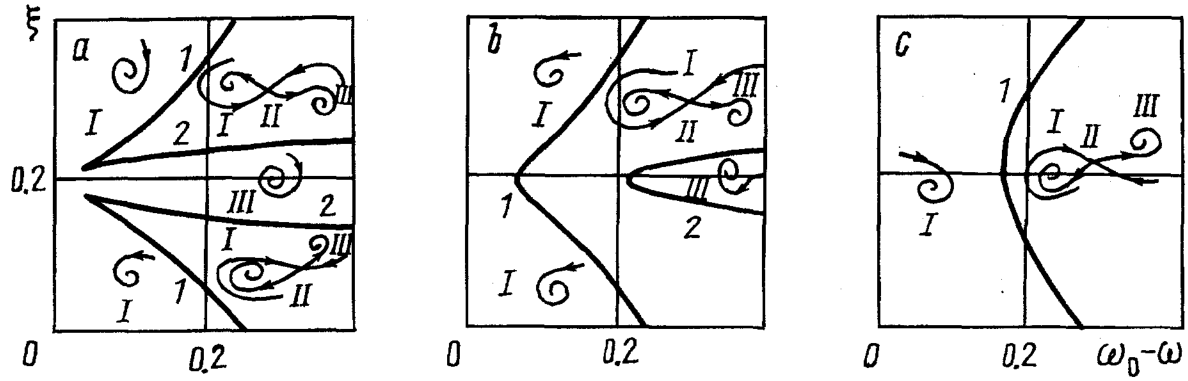


Fig. 1. Bifurcation diagrams and phase trajectories (24) for an oscillating vapor bubble under different heat and mass transfer conditions [1) (29); 2) (30); I, II) stable and unstable resonance; III) stable nonresonance]: a) $M_1 = 0.055$, $M_2 = 0$; b) $M_1 = 0.05$, $M_2 = 0.01$; c) $M_1 = 0.05$, $M_2 = 0.03$.

If the vibrations of the bubble are stationary, then $dc/d\tau = 0$, $d\varphi/d\tau = 0$. Then, equating to zero the left-hand sides of Eqs. (24), we can write the expression for c:

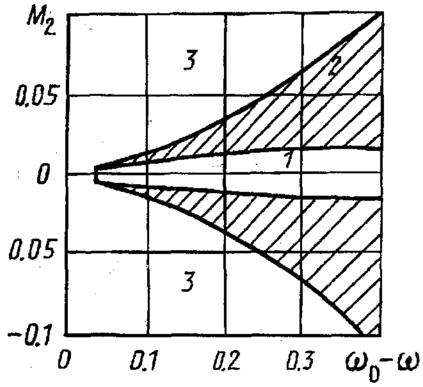
$$c^6 + c^4 \frac{4\omega_0}{N}(\omega - \omega_0) + \frac{4c^2\omega_0^2}{N^2}\left[b^2 + (\omega - \omega_0)^2\right] - \frac{\xi^2}{N^2} - \frac{4\omega_0}{N^2}\left(M_1^2 + M_2^2 - \frac{\xi}{\omega_0} M_1\right) = 0. \tag{27}$$

The phase shift for the stationary vibrations is

$$\varphi = \arcsin\left[-\frac{\dfrac{M_2\left[(\omega_0 - \omega)c - \dfrac{N}{2\omega_0}c^3\right]}{\xi/2\omega_0 - M_1} + bc}{\dfrac{\xi}{2\omega_0} - M_1 + \dfrac{M_2^2}{\xi/2\omega_0 - M_1}}\right]. \tag{28}$$

Let us analyze qualitatively the change in the character of bubble oscillations as a function of amplitude and frequency of the external effect in the main resonance region. For this purpose, we consider a bubble with the initial radius $R_0 = 10^{-5}$ m. We shall tentatively assume that $\rho = 998$ kg·m^{-3}, $\sigma = 0.0725$ N·m^{-1}, $\mu = 0.001$ kg·m^{-1}·sec^{-1}, $P_\infty^* = 101{,}300$ Pa, $\gamma = 4/3$. The processes of heat and mass transfer on the oscillating bubble walls were specified by the parameters M_1 and M_2.

The results obtained are presented in Fig. 1 where three characteristic bifurcation diagrams and the corresponding phase trajectories are depicted. To obtain the phase trajectories, the system of equations (24) was solved by the method of numerical integration. We fix the value $\omega - \omega_0 = 0.2$ (Fig. 1a). It can easily be seen that if the value of ξ is smaller than the corresponding value on curve 1 ($\xi < \xi_1$), then one type of equilibrium exists in the oscillating bubble system: a stable focus (see the phase trajectory), where the bubble performs stable resonance vibrations. As soon as $\xi = \xi_1$, saddle-node bifurcation takes place, as a result of which three singular points appear in the system: two stable focuses and a saddle. Here, stable and nonstable resonance oscillations take place (points I and II), as well as stable nonresonance bubble oscillations III. With a further increase in ξ, we already intersect curve 2 ($\xi = \xi_2$). Here, the reverse bifurcation occurs, as a result of which one type of equilibrium exists again in the system - a stable focus, but here the vibrations have a nonresonance character. The second intersection with curve 2 (upper branch) is again accompanied by a saddle-node bifurcation. And when we intersect curve 1 for the last time, instead of three singular points one stable focus exists again in the system as a result of the reverse bifurcation. Thus, in the system of an oscillating vapor bubble in the case of monotonic change in the amplitude of the external effect,



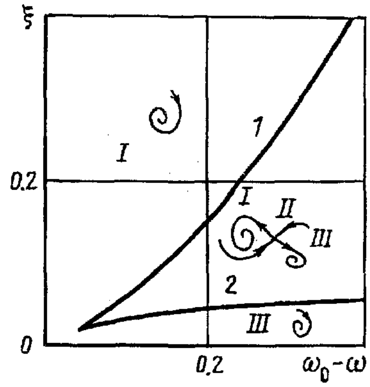


Fig. 2. Diagram of various system states as a function of M_2: 1) region of multiple saddle-node bifurcation; 2) region of single saddle-node bifurcation; 3) region of no saddle-node bifurcation.

Fig. 3. Bifurcation diagram and phase trajectories (24) for an oscillating gas bubble ($M_1 = M_2 = 0$): 1) (29); 2) (30); I, II) stable and unstable resonance; III) stable nonresonance.

one can observe direct and reverse saddle-node bifurcation many times. The equations for curves 1 and 2 are obtained from Eq. (27). They have the forms

$$\xi_1 = 2\omega_0 M_1 \pm \left\{\frac{16\omega_0^3}{3N}(\omega_0 - \omega)\left[b^2 + \frac{(\omega - \omega_0)^2}{9}\right] + \right.$$
$$\left. + \frac{16\omega_0^3}{N}\left[\frac{(\omega_0 - \omega)^2}{9} - \frac{b^2}{3}\right]^{3/2} - 4\omega_0^2 M_2^2\right\}^{1/2}, \tag{29}$$

$$\xi_2 = 2\omega_0 M_1 \pm \left\{\frac{16\omega_0^3}{3N}(\omega_0 - \omega)\left[b^2 + \frac{(\omega - \omega_0)^2}{9}\right] - \right.$$
$$\left. - \frac{16\omega_0^3}{N}\left[\frac{(\omega_0 - \omega)^2}{9} - \frac{b^2}{3}\right]^{3/2} - 4\omega_0^2 M_2^2\right\}^{1/2}. \tag{30}$$

The shift in frequency to the resonance side leads to the disappearance of the region with three singular points. Thus, when $\omega - \omega_0 < 0.03$ (Fig. 1a), one type of equilibrium exists in the system irrespective of the value of ξ, viz., a stable focus (resonance vibrations), and saddle-node bifurcation is not observed.

The character of the bifurcation diagram is determined in many respects by the value of M_2. It is seen from Fig. 1b that within a certain range of frequencies saddle-node bifurcation and reverse bifurcation occur once with a monotonic change in ξ, i.e., there is no curve 2 in the given region. However, in moving away from the natural frequency ω_0, a similar replacement of equilibrium states is observed in the system, just as in Fig. 1a. These conditions are also possible for the occurrence of heat and mass transfer on the bubble wall, when saddle-node and reverse bifurcations occur only once in the region of the main resonance with a change in the amplitude of the external effect (Fig. 1c).

It is seen from Fig. 2 that as the value of $\omega_0 - \omega$ increases, the regions of single and multiple saddle-node bifurcation grow.

For comparison with the results obtained, we consider a gas bubble. Let the parameters of the oscillating gas bubble system be the same as for a vapor bubble, but there are no heat and mass exchange processes ($M_1 = M_2 = 0$). In Fig. 3 the bifurcation diagram for an oscillating gas bubble in an incompressible fluid is presented. As is seen from Fig. 3, starting with $\omega - \omega_0 > 0.035$, saddle-node bifurcation and reverse bifurcation may occur in the system. But with a monotonic change in the amplitude of the external effect, this process is observed only once. In fact, when ξ increases, the stable focus (nonresonance oscillations) is replaced by two stable focuses and a saddle as a result of saddle-node bifurcation as soon as $\xi > \xi_2$. A further increase in ξ is accompanied by reverse bifurcation at the intersection of curve 1, where the system develops the stable-focus type of equilibrium (resonance oscillations). A similar single character of saddle-node bifurcation is observed for an oscillating gas bubble in a compressible fluid [8].

Based on the results presented above, it is possible to determine the character of the change in the bubble vibration parameters, namely, the amplitude and phase, with a change in the amplitude of the external effect on the bubble. Thus, knowing the dynamics of the change in the amplitude and phase, we may determine the character of the vapor bubble oscillations at any instant.

In conclusion it should be noted that the considered mechanism of the replacement of the stable states of a vapor bubble in the main resonance region is the initial stage in the transition from regular to random oscillations.

NOTATION

R, R_0, instantaneous and equilibrium radii of the bubble; τ, time; ρ', ρ, densities of the vapor and liquid phases; σ, surface tension coefficient; μ, viscosity; Ω, frequency of the external effect; ω_0, dimensionless natural frequency of the system; P^*_∞, constant pressure; P_a, amplitude of the external effect amplitude; γ, polytrope index; ρ_0, constant density of the vapor; ω, dimensionless frequency of the external effect.

REFERENCES

1. R. I. Nigmatulin, Dynamics of Multiphase Media [in Russian], Moscow (1987).
2. N. A. Gumerov, Prikl. Mat. Mekh., **55**, No. 2, 256-263 (1991).
3. S. N. Syromyatnikov, in: Kinetic Theory of Transfer Processes in Evaporation and Condensation Transfer Processes (Proc. of the Int. School-Seminar), Minsk (1991), pp. 51-56.
4. A. Naife, Introduction to Pertubation Methods [Russian translation], Moscow (1984).
5. L. Samek, Czech. J. Phys. Bd., **39**, No. 12, 1354-1371 (1989).
6. V. A. Akulichev, V. N. Alekseyev, and V. A. Bulanov, Periodic Phase Changes in Fluids [in Russian], Moscow (1986).
7. U. Pollman, Acustica, **68**, No. 4, 241-250 (1989).
8. P. Smereka, B. Birnir, and S. Banerjee, Phys. Fluids, **30**, No. 11, 3342-3350 (1987).